

Resonance saturation of the chiral couplings at NLO in $1/N_C$

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The precision obtainable in phenomenological applications of Chiral Perturbation Theory is currently limited by our lack of knowledge on the low-energy constants (LECs). The assumption that the most important contributions to the LECs come from the dynamics of the low-lying resonances, often referred to as the resonance saturation hypothesis, has stimulated the use of large- N_C resonance lagrangians in order to obtain explicit values for the LECs. We study the validity of the resonance saturation assumption at the next-to-leading order in the $1/N_C$ expansion within the framework of Resonance Chiral Theory (R χ T). We find that, by imposing QCD short-distance constraints, the chiral couplings can be written in terms of the resonance masses and couplings and do not depend explicitly on the coefficients of the chiral operators in the Goldstone boson sector of R χ T. As we argue, this is the counterpart formulation of the resonance saturation statement in the context of the resonance lagrangian. Going beyond leading order in the $1/N_C$ counting allow us to keep full control of the renormalization scale dependence of the LEC estimates.

PACS numbers: 11.15.Pg, 12.38.-t, 12.39.Fe

I. INTRODUCTION

Chiral Perturbation Theory (χ PT) is the effective field theory of Quantum Chromodynamics (QCD) at very-low energies [1]. Theoretical calculations in this framework have nowadays reached the two-loop level accuracy. The predictions from χ PT are parameterized in terms of the $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ low-energy constants (LECs), which are not fixed by the chiral symmetry. Given the large uncertainties in the $\mathcal{O}(p^4)$ coupling estimates [2], and the poor knowledge and huge number of $\mathcal{O}(p^6)$ couplings (90 in $SU(3)$ and 53 in $SU(2)$ [3], just in the even intrinsic parity sector), a proper determination of LECs turns into a difficult task. Actually, one of the major problems for doing phenomenology using χ PT at the two-loop level comes from our ignorance on these constants [4].

Different theoretical approaches to determine the chiral couplings have been pursued recently (see for instance Ref. [5] for the state of the art in the context of lattice QCD or the recent work [6] in the framework of QCD sum-rules). Among the latter, the use of large- N_C lagrangians has been proven very fruitful [7, 8, 9, 10]. By integrating out the resonances fields, one can obtain the LECs parameterized in terms of resonance masses and couplings. The method relies on the assumption that the most important contribution to the LECs come from the physics of the low-lying resonances.

Resonance Chiral Theory (R χ T) [7, 8, 9] provides a chiral invariant lagrangian description of the meson interactions in the resonance region, $M_\rho \lesssim E \lesssim 2$ GeV, where the main problems to develop a formal effective field theory are the existence of many resonances with close masses and the apparent absence of a natural expansion parameter. The phenomenological lagrangian is nevertheless ruled by the $1/N_C$ expansion [11]: tree-level

interactions between the large- N_C infinite spectrum of narrow states provide the leading order (LO) contribution to Green functions of QCD currents, being the next-to-leading order (NLO) corrections given by one-loop diagrams and subleading tree contributions

The Green functions generated from the resonance lagrangian must obey QCD short-distance constraints. The matching between Green functions evaluated from R χ T [10] or constructed using general meromorphic functions with resonance poles [12, 13, 14, 15] and the asymptotic behaviour dictated by QCD yields valuable information on the resonance couplings. In this way Green functions describing the resonance region are used as a bridge between QCD and ChPT, allowing the determination of LECs in terms of a few hadronic parameters. This matching procedure has been realized at the practical level by approximating the hadronic spectrum to a finite number of states, thus introducing a model-dependence in the description. The truncation of the tower and the choice of an appropriate set of short-distance constraints for each case constitute the so-called Minimal Hadronic Approximation [12], which can be implemented in an equivalent way by using general meromorphic functions or a chiral resonance lagrangian. An implicit hypothesis in the predictions obtained from R χ T is that they should approach the actual values of the QCD low-energy constants as more and more resonances are added to the theory. This hypothesis is non-trivial and has been a subject of investigation in Refs. [15, 16, 17]. In particular, possible conflicts with the short-distance matching have been pointed out in Ref. [17].

A systematic study of the large- N_C determinations of the $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ χ PT couplings within R χ T has been undertaken in Refs. [7] and [9], respectively. A well-known drawback of the estimates obtained upon integra-

tion of the resonances at tree-level is that we are unable to control their renormalization scale dependence, being the latter a subleading effect in $1/N_C$. Typically, these LO predictions are assumed to correspond to a value $\mu \sim M_\rho$, but a large uncertainty from variations of the scale is unavoidable. Clearly, this arbitrariness on the choice of the scale for the LEC estimates disappears if the matching between χ PT and $R\chi$ T is done at the next-to-leading order in $1/N_C$ in both theories (*i.e.* including loops with mesons), as it has been corroborated in recent attempts to determine the LECs at NLO within $R\chi$ T [18, 19, 20, 21].

The estimate for the constant L_i of a χ PT operator \mathcal{O}_i obtained from $R\chi$ T depends on an equivalent (in principle unknown) constant \tilde{L}_i , corresponding to the coupling of an operator with the same structure as \mathcal{O}_i , but living in the theory where the resonances are active degrees of freedom. At LO, it was found in Ref. [8] that the couplings \tilde{L}_i corresponding to operators of chiral order $\mathcal{O}(p^4)$ were fixed in terms of resonance couplings and masses once short-distance QCD constraints were imposed in the effective lagrangian. In particular, they were found to be zero in the antisymmetric tensor formalism for spin-1 fields [8]. Therefore, upon tree-level integration of the resonances, one obtains predictions for the χ PT low-energy couplings in terms of just resonance parameters related to operators which involve resonance fields. The later statement provides a precise definition of what should be understood by *resonance saturation* in the context of $R\chi$ T. It is the aim of this work to show that the statement also holds when NLO corrections in $1/N_C$ in $R\chi$ T are considered: the \tilde{L}_i get as well fixed in terms of resonance parameters and, consequently, the prediction for the LECs only depend on the values of the resonance couplings and masses. This result has been used implicitly in recent works [20, 21], where NLO estimations of the constants L_8 , C_{38} and L_{10} , C_{87} were extracted from the analysis of the $SS - PP$ and the $VV - AA$ correlators, respectively. The present paper is intimately related to Ref. [22], which was devoted to the same subject considering a $R\chi$ T lagrangian with just Goldstones, scalar and pseudoscalar resonances [23]. Here we provide the general proof including also vector and axial-vector mesons, thus clarifying the role of resonance saturation in a wider range of applications.

The outline of the paper is as follows. In Section II we introduce the aspects of χ PT and $R\chi$ T which are relevant for our case, focusing on the structure of the lagrangians and their related couplings and power-counting. In Section III we discuss the precise meaning of resonance saturation in the framework of $R\chi$ T, and give specific examples at leading and next-to-leading order in $1/N_C$. In Section IV we prove that resonance saturation is fulfilled for those LECs related to QCD amplitudes obeying high-energy constraints. The demonstration will be based on a careful analysis of the analytic structure of the matrix elements calculated with $R\chi$ T up to NLO. Finally, Section V summarizes our results.

II. CHIRAL RESONANCE LAGRANGIAN

Chiral Perturbation Theory is organized as a perturbative expansion in powers of light quark masses and derivatives of the Goldstone fields [1],

$$\mathcal{L}_{\chi PT} = \sum_{n \geq 1} \mathcal{L}_{2n}^{\chi PT}, \quad (1)$$

with $\mathcal{L}_{2n}^{\chi PT} \sim \mathcal{O}(p^{2n})$. The leading-order term

$$\mathcal{L}_2^{\chi PT} = \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle \quad (2)$$

contains only two couplings, the meson decay constant in the chiral limit F and the constant B_0 inside $\chi_+ \sim m_q B_0$, related to the quark condensate. The chiral tensor u_μ contains a derivative acting on the Goldstone fields so it is of order p in the chiral counting. The tensor $\chi_+ \sim \mathcal{O}(M_\pi^2)$ counts as order p^2 in the standard formulation of χ PT where the light quark mass and the momenta are related. The low-energy constants in the effective lagrangian are not fixed by symmetry requirements and their number increases quickly with the chiral order. At $\mathcal{O}(p^4)$ ten additional couplings are allowed by the chiral symmetry:

$$\begin{aligned} \mathcal{L}_4^{\chi PT} = & L_1 \langle u_\mu u^\mu \rangle^2 + L_2 \langle u_\mu u^\nu \rangle \langle u^\mu u_\nu \rangle + L_3 \langle u_\mu u^\mu u_\nu u^\nu \rangle \\ & + L_4 \langle u_\mu u^\mu \rangle \langle \chi_+ \rangle + L_5 \langle u_\mu u^\mu \chi_+ \rangle + L_6 \langle \chi_+ \rangle^2 \\ & + L_7 \langle \chi_- \rangle^2 + L_8/2 \langle \chi_+^2 + \chi_-^2 \rangle - iL_9 \langle f_+^{\mu\nu} u_\mu u_\nu \rangle \\ & + L_{10}/4 \langle f_{+\mu\nu} f_+^{\mu\nu} - f_{-\mu\nu} f_-^{\mu\nu} \rangle, \end{aligned} \quad (3)$$

where the $SU(3)$ case has been considered and we have dismissed contact terms and operators that vanish when the equations of motion are used. We have showed explicitly the form of the $\mathcal{O}(p^4)$ chiral structures to identify the relevant Green functions to which the LECs L_i contribute. Since the vector, axial-vector, scalar and pseudoscalar sources are contained in the chiral tensors $f_+^{\mu\nu}$, $f_-^{\mu\nu}$, χ_+ and χ_- , respectively, and u_μ involves at least one Goldstone boson, it follows that at $\mathcal{O}(p^4)$ in the chiral limit: (i) L_1 , L_2 and L_3 determine the Goldstone boson scattering, (ii) L_4 and L_5 the scalar form factor of the pion, (iii) $L_6 + L_7$ and L_8 the difference of the scalar and pseudoscalar correlators, (iv) L_6 the two-point Green function of two scalar densities $\bar{q}q$ and $\bar{q}'q'$ with $q \neq q'$, (v) L_9 the vector form factor of the pion, and (vi) L_{10} the difference of the two-point correlation function of vector and axial-vector currents.

For $\mathcal{O}(p^6)$ accuracy we have to account for 90 new independent terms only in the even intrinsic parity sector with coefficients C_i ,

$$\mathcal{L}_6^{\chi PT(\text{even})} = \sum_{i=1}^{90} C_i \mathcal{O}_i^{(6)}. \quad (4)$$

From an effective field theory point of view, the Goldstone interactions at low-energies are affected by the dynamics of hadronic states of higher masses (resonances),

which have been integrated out. These effects can be studied systematically with the help of a lagrangian description of the chiral invariant Goldstone-resonance interactions which takes the $1/N_C$ expansion as a guiding principle, as described in Refs. [7, 8, 9]. The lagrangian of Resonance Chiral Theory can be organized according to the number of resonance fields in the interaction terms,

$$\mathcal{L}_{\text{R}\chi\text{T}} = \mathcal{L}^{\text{GB}} + \mathcal{L}^{R_i} + \mathcal{L}^{R_i R_j} + \mathcal{L}^{R_i R_j R_k} + \dots, \quad (5)$$

where R_i stands for resonance multiplets of vectors $V(1^{--})$, axial-vectors $A(1^{++})$, scalars $S(0^{++})$ and pseudoscalars $P(0^{-+})$. In order to carry out the matching of $\text{R}\chi\text{T}$ and χPT , one is forced to truncate the infinite tower of resonances of the large- N_C spectrum to a finite amount of them. As more hadrons are added to the theory, the $\text{R}\chi\text{T}$ amplitudes are expected to approach progressively its actual large- N_C value [12, 13]. The arguments that will be used to establish resonance saturation in the next sections can be easily generalised to any number of multiplets. To simplify the discussion we will nevertheless assume in the following a large- N_C lagrangian $\mathcal{L}_{\text{R}\chi\text{T}}$ truncated to the lowest-lying resonance multiplet in each channel.

The term $\mathcal{L}^{\text{GB}} = \mathcal{L}_2^{\text{GB}} + \mathcal{L}_4^{\text{GB}} + \dots$ in Eq. (5) is the Goldstone chiral lagrangian, which shares the same general operator structure as the χPT lagrangian (1). However, the values of the \mathcal{L}^{GB} couplings ($\tilde{L}_i, \tilde{C}_i \dots$) are different from the χPT LECs ($L_i, C_i \dots$), since the Goldstone self-interactions in the presence of resonances differ from those at low-energies where the resonances have been integrated out. Therefore a new set of (a priori unknown) constants is introduced in the Goldstone sector of the resonance lagrangian.

Adopting the antisymmetric tensor formalism to describe the spin-1 resonance fields, the second term in Eq. (5) reads [7]

$$\begin{aligned} \mathcal{L}^{R_i} &= \mathcal{L}^V + \mathcal{L}^A + \mathcal{L}^S + \mathcal{L}^P, \\ \mathcal{L}_{(2)}^V &= \frac{F_V}{2\sqrt{2}} \langle V_{\mu\nu} f_+^{\mu\nu} \rangle + \frac{i G_V}{2\sqrt{2}} \langle V_{\mu\nu} [u^\mu, u^\nu] \rangle, \\ \mathcal{L}_{(2)}^A &= \frac{F_A}{2\sqrt{2}} \langle A_{\mu\nu} f_-^{\mu\nu} \rangle, \\ \mathcal{L}_{(2)}^S &= c_d \langle S u_\mu u^\mu \rangle + c_m \langle S \chi_+ \rangle, \\ \mathcal{L}_{(2)}^P &= i d_m \langle P \chi_- \rangle, \end{aligned} \quad (6)$$

where only the terms with a chiral tensor of $\mathcal{O}(p^2)$ have been shown. The lagrangian (6), supplemented with the kinetic and mass terms for the resonance fields, yields the most general lagrangian that can give contributions to the $\mathcal{O}(p^4)$ LECs after integrating out the resonances at tree-level (*i.e.* at LO in the large- N_C expansion) [7, 8]. The latter is easily understood because integrating out a resonance field R_i yields a series of chiral monomials starting at $\mathcal{O}(p^2)$. Thus for the chiral $\mathcal{O}(p^6)$ LECs, the

terms in \mathcal{L}^{R_i} , $\mathcal{L}^{R_i R_j}$ and $\mathcal{L}^{R_i R_j R_k}$ with chiral tensors of order p^4 , p^2 and p^0 , respectively, are required (the complete basis of operators can be found in Ref. [9]). This counting does not necessarily apply at NLO in $1/N_C$ because resonances occur inside loops. The loop integration can trade off powers of p^2 for resonance masses, so that the low-energy expansion of such contributions can yield monomials in p^2 of lower order than expected by dimensional analysis. Thus resonance interactions with higher derivatives that do not contribute to $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ LECs upon integration at tree-level can become relevant when loop effects are taken into account. Since our analysis concerns the determination of the LECs including one-loop contributions in the resonance chiral theory, we shall formally consider resonance terms in $\mathcal{L}_{\text{R}\chi\text{T}}$ with chiral tensors of arbitrary order. Nevertheless, for most phenomenological applications the interactions with large number of derivatives in Eq. (5) can be shown to violate the QCD ruled asymptotic behavior of Green functions and form factors at high energies, and thus get severely restricted.

III. RESONANCE SATURATION IN $\text{R}\chi\text{T}$

The notion of resonance saturation has been vaguely used in the literature of low-energy QCD to designate a number of cases where the strong interactions are essentially described by meson-resonance exchanges. In the framework of the large- N_C -inspired lagrangian of Eq. (5), resonance saturation is linked to the estimation of the chiral LECs from the knowledge of the resonance parameters. In particular, it is related to the role of the couplings in the Goldstone sector \mathcal{L}^{GB} , as we argue next.

Upon integration of the resonances one gets an expression for any chiral coupling in terms of the parameters in the $\text{R}\chi\text{T}$ lagrangian:

$$\begin{aligned} L_i &= \tilde{L}_i + f_i(M_R, \alpha_R), \\ C_i &= \tilde{C}_i + g_i(M_R, \alpha_R), \dots \end{aligned} \quad (7)$$

where $f_i(M_R, \alpha_R)$ and $g_i(M_R, \alpha_R)$ are the contribution stemming from the low-energy expansion of the resonance contributions (*i.e.* from the diagrams that contain resonance lines), which include one-loop diagrams if we work at NLO in $1/N_C$. M_R denotes generically the resonance masses while α_R stands for the $\text{R}\chi\text{T}$ couplings accompanying operators with resonance fields. Clearly, Eq. (7) is useless for determining the LECs if the couplings \tilde{L}_i in the Goldstone boson sector of $\mathcal{L}_{\text{R}\chi\text{T}}$ remain unknown parameters. We shall state that resonance saturation is fulfilled in the matching between $\mathcal{L}_{\text{R}\chi\text{T}}$ and $\mathcal{L}_{\chi\text{PT}}$ if the \tilde{L}_i couplings get fixed completely by the short-distance constraints, so that the L_i are then given as functions of only M_R and α_R . This definition implies that the saturation is accomplished for any value of the χPT renormalization scale μ (the “extreme” version of resonance saturation pointed out in Ref. [18]).

The condition of resonance saturation was confirmed at leading-order in $1/N_C$ for the $\mathcal{O}(p^4)$ LECs in Refs. [7, 8]. In the $R\chi T$ formulation where spin-1 resonances are described by antisymmetric tensor fields it was found that the \tilde{L}_i actually vanish due to short-distance constraints and Eq. (7) turns out to be

$$\begin{aligned} L_1 &= \frac{G_V^2}{8M_V^2}, \quad L_2 = \frac{G_V^2}{4M_V^2}, \quad L_3 = -\frac{3G_V^2}{4M_V^2} + \frac{c_d^2}{2M_S^2}, \\ L_5 &= \frac{c_d c_m}{M_S^2}, \quad L_8 = \frac{c_m^2}{2M_S^2} - \frac{d_m^2}{2M_P^2}, \quad L_9 = \frac{F_V G_V}{2M_V^2}, \\ L_{10} &= -\frac{F_V^2}{4M_V^2} + \frac{F_A^2}{4M_A^2}, \quad L_4 = L_6 = L_7 = 0, \end{aligned} \quad (8)$$

that is, one is able to determine the $\mathcal{O}(p^4)$ chiral couplings of Eq. (3) in terms of the resonance parameters in Eq. (6). For the $\mathcal{O}(p^6)$ LECs C_i a complete analysis of the saturation hypothesis has not been yet performed but it has often been assumed to hold in phenomenological applications. (For a recent review of the status of the LEC determinations from $R\chi T$ see Ref. [24] and references therein).

As an illustrative example of how the saturation is fulfilled at leading order in $1/N_C$, let us consider the two-point correlation functions of two vector or axial-vector currents in the chiral limit. Of particular interest is their difference $\Pi(q^2) \equiv \Pi_{VV}(q^2) - \Pi_{AA}(q^2)$. At large N_C , the resonance lagrangian of Eq. (5) produces for the $VV-AA$ charged correlator [21]

$$\Pi(q^2) = \frac{2F^2}{q^2} + \frac{2F_V^2}{M_V^2 - q^2} - \frac{2F_A^2}{M_A^2 - q^2} - 8\tilde{L}_{10} + 16\tilde{C}_{87} q^2 + \dots \quad (9)$$

being q the momentum flowing into the current vertex. In Eq. (9) we have explicitly shown contributions from \mathcal{L}^{GB} up to chiral order p^6 . From QCD we know that this correlator vanishes when $q^2 \rightarrow \infty$ [25]. Imposing this behavior in Eq. (9), one gets that $\tilde{L}_{10} = \tilde{C}_{87} = 0$. The same correlator computed at low-energies with the χPT lagrangian gives

$$\Pi^{\chi\text{PT}}(q^2) = \frac{2F^2}{q^2} - 8L_{10} + 16C_{87} q^2 + \dots \quad (10)$$

Comparing this result with the low-energy expansion of Eq. (9), one is able to estimate the corresponding chiral couplings as a function of the resonance masses and couplings only:

$$L_{10} = -\frac{F_V^2}{4M_V^2} + \frac{F_A^2}{4M_A^2}, \quad C_{87} = \frac{F_V^2}{8M_V^4} - \frac{F_A^2}{8M_A^4}. \quad (11)$$

Note that the important point for the resonance saturation condition to hold is that the value of the corresponding \mathcal{L}^{GB} couplings have been fixed by requiring the right short-distance behaviour of the Green function computed with the resonance theory. A fixed but non-zero value for the \mathcal{L}^{GB} couplings can be required for consistency with

QCD if a different representation for the spin-1 resonance fields (the Proca formalism, for example) is chosen [8]. A similar situation has also been found in a recent analysis of spin-2 resonances [26], which shows that some of the coefficients in \mathcal{L}^{GB} must be different from zero, though the χPT LECs get finally determined in terms of resonance parameters only.

The resonance saturation condition applies also at next-to-leading order in the $1/N_C$ expansion. We consider again the case of the $VV-AA$ correlator $\Pi(q^2)$ as an example. Its expression up to NLO within $R\chi T$ can be split in the following way:

$$\Pi(q^2) = \Pi^{\text{'good'}}(q^2) + \Pi^{\text{'bad'}}(q^2) + \Pi^{\text{GB}}(q^2). \quad (12)$$

The term $\Pi^{\text{'good'}}(q^2)$ collects the part of the amplitude which vanishes at high energies, while $\Pi^{\text{'bad'}}(q^2)$ and $\Pi^{\text{GB}}(q^2)$ collect the pieces which grow as $\mathcal{O}(q^0)$ or faster for large q^2 . The piece $\Pi^{\text{GB}}(q^2)$ arises from the local contributions of \mathcal{L}^{GB} ,

$$\Pi^{\text{GB}}(q^2) = -8\tilde{L}_{10} + 16\tilde{C}_{87} q^2 + \dots \quad (13)$$

and only one-loop diagrams contribute to the piece $\Pi^{\text{'bad'}}(q^2)$, since the tree-level meson exchanges already satisfy the short-distance constraints. Resonance saturation states that the local terms $\tilde{L}_i, \tilde{C}_i \dots$ get fixed once the correct asymptotic behavior is imposed in the theory. In the case at hand, the short-distance constraint implies that

$$\Pi^{\text{'bad'}}(q^2) + \Pi^{\text{GB}}(q^2) = 0, \quad (14)$$

and consequently

$$\Pi(q^2) = \Pi^{\text{'good'}}(q^2). \quad (15)$$

From the low-energy expansion of Eq. (15) one can extract estimates for the renormalized chiral couplings L_{10}^r, C_{87}^r with NLO accuracy, thus keeping control of their one-loop renormalization scale dependence: the matching equations between $R\chi T$ and χPT read

$$\begin{aligned} -8L_{10}^r(\mu) &= \lim_{q^2 \rightarrow 0} \left[\Pi^{\text{'good'}}(q^2) - \hat{\Pi}^{\chi\text{PT}}(q^2; \mu) \right], \\ 16C_{87}^r(\mu) &= \lim_{q^2 \rightarrow 0} \left[\frac{d}{dq^2} \left(\Pi^{\text{'good'}}(q^2) - \hat{\Pi}^{\chi\text{PT}}(q^2; \mu) \right) \right], \end{aligned} \quad (16)$$

where $\hat{\Pi}^{\chi\text{PT}}$ is the $VV-AA$ correlator in χPT [1] without the local contributions $L_{10}^r, C_{87}^r \dots$, which have been isolated on the l.h.s. of the equations above. The cancellation of the $\ln(-q^2)$ terms in the differences in the r.h.s of Eqs. (16) is ensured because $R\chi T$ reproduces χPT at low energies. As a consequence, the limits are well defined for $q^2 \rightarrow 0$ and the μ dependence of $\hat{\Pi}^{\chi\text{PT}}(q^2; \mu)$ gives the right renormalization scale running of the χPT LECs.

The solution of Eq. (14) for the \mathcal{L}^{GB} couplings requires that $\Pi^{\text{'bad'}}(q^2)$ does not have non-polynomial terms (like

$\log(-q^2)$). In Section IV A, we show that for a general two-point correlator of two currents the non-polynomial terms can be made to vanish by a suitable set of the tree-level resonance parameters. To extend the saturation condition to the LECs related to the pion form factors and to Goldstone boson scattering one has to impose in an analogous way that the parts with the wrong asymptotic behaviour in the corresponding amplitudes can be made free of non-polynomial terms. Sections IV B and IV C deal with the proof for the pion form factors and for Goldstone scattering, respectively.

Resonance saturation at NLO with a resonance Lagrangian involving only scalars and pseudoscalars mesons was discussed in Ref. [22]. There it was found that for those LECs which get tree-level contributions from scalar and pseudoscalar resonance exchange (namely L_{4-8}), the corresponding local couplings in $R\chi T$ get fixed to $\tilde{L}_i = 0$. This more restrictive condition arises because the analysis in Ref. [22] proved that the short-distance conditions on the SS and PP correlators yield $\Pi^{\text{bad}}(q^2) = 0$, so that $\tilde{L}_i = \tilde{C}_i = 0$ follow from Eq. (14). A key ingredient in the proof of Ref. [22] was that the spin-0 resonance propagator behaves as $\mathcal{O}(1/q^2)$ for large q^2 . In this work we address the more general case which accounts also for spin-1 resonance fields. They are conventionally described in $R\chi T$ in the antisymmetric tensor field formalism though other formalisms can be used [7, 8]. Contrary to the spin-0 case, the propagator of a massive spin-1 particle scales like $\mathcal{O}(q^0)$ for $q^2 \rightarrow \infty$. Recall, for instance, the form of the propagator in the Proca formalism,

$$\Delta^{\mu\nu}(q^2) = \frac{-i}{q^2 - M_R^2} \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{M_R^2} \right). \quad (17)$$

Similarly, the antisymmetric field propagator contains a piece that does not fall off as $\mathcal{O}(1/q^2)$ for $q^2 \rightarrow \infty$ (see Ref. [7] for the explicit expression). Therefore, the conclusions obtained for scalar and pseudoscalar resonances by inspection of the large q^2 behaviour of the one-loop amplitudes in Ref. [22] do not translate trivially to the case of vectors and axial-vector resonances, which require an independent analysis.

IV. QCD AMPLITUDES AT NLO IN $1/N_C$

In this section we analyze the asymptotic behaviour for large momenta of the amplitudes which are relevant to fix the \mathcal{L}^{GB} couplings through conditions analogous to Eq. (14). In particular, we prove that non-polynomial terms do not appear in the latter conditions when the amplitudes obey the short-distance constraints.

The leading-order term in the large- q^2 expansion of a $R\chi T$ one-loop amplitude is obtained easily by simple dimensional analysis. The large- q^2 counting rules are that any vertex with a chiral tensor of order $p^{(2n)}$ yields at most q^{2n} . Spin-0 and spin-1 propagators outside the

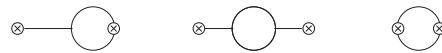


FIG. 1: Topologies in the one-loop correlators with an intermediate two-meson cut. The lines can represent both Goldstone and resonance fields.

loop¹ count as $(q^2)^{-1}$, while a spin-1 propagator inside a loop is $\mathcal{O}(q^0)$ (see Eq. (17)). The loop integration measure gives $\int d^4k \sim q^4$. The chiral tensors containing the external currents (*i.e.* χ_\pm and $f_\pm^{\mu\nu}$) can be booked as $\mathcal{O}(q^0)$ once we drop the momentum tensor structure introduced by the vector and axial-vector currents. As an example, the tree-level exchange of a vector resonance in the correlator of two vector currents (Π_{VV}) counts as $1/q^2$ according to the rules, which is clear from the explicit result:

$$\Pi_{VV}(q^2) = \frac{2F_V^2}{M_V^2 - q^2}. \quad (18)$$

One-loop corrections to Π_{VV} arise from diagrams as those shown in Fig. 1. Let us assume that all vertices in the diagrams contain a chiral tensor of $\mathcal{O}(p^2)$. If there are no spin-1 resonances running inside the loops then the diagrams behave as $\mathcal{O}(q^0)$ for large q^2 . If we allow for one or two spin-1 resonances in the loops then the diagrams are up to $\mathcal{O}(q^2)$ or $\mathcal{O}(q^4)$, respectively.

We analyse in what follows the large- q^2 structure of the two-current correlators, pion form factors and Goldstone scattering amplitude separately. To simplify the expressions we will consider only resonance operators in $\mathcal{L}^{R_i}, \mathcal{L}^{R_i R_j} \dots$ with a chiral tensor up to $\mathcal{O}(p^2)$. The generalization of our findings for the case of higher-order interaction terms would be straightforward.

A. Two-point correlators

Let us consider the two-point functions built from two scalar (SS) or pseudoscalar (PP) densities, or vector (VV) or axial-vector (AA) currents. Their tree-level expressions are given by one-particle exchanges, so they are booked as $\mathcal{O}(q^{-2})$ at large energies according to the counting explained above. On the other hand, the one-loop diagrams can give up to $\mathcal{O}(q^4)$ terms if we allow for spin-1 mesons in the absorptive part. After reduction to scalar integrals, all one-loop terms are proportional to the scalar two- and one-point functions $B_0(q^2, M_i^2, M_j^2)$ and $A_0(M_i^2)$ [27]. Expanding out the coefficients in front of

¹ When the vector (axial-vector) resonance is connected to an external vector (axial-vector) current, the part of the spin-1 propagator which behaves as $\mathcal{O}(q^0)$ for large q^2 does not contribute because of the structure of the vertices. The same holds when the vector resonance is coupled to a pair of Goldstone bosons.

the scalar integrals for $q^2 \rightarrow \infty$, the one-loop amplitude has the form [16, 22]

$$\begin{aligned} \Pi^{1-\text{loop}}(q^2) = & \sum_n B_0(q^2, M_i^2, M_j^2) \left(\lambda_n^{(0)} q^0 + \lambda_n^{(2)} q^2 + \lambda_n^{(4)} q^4 \right) \\ & + \sum_\ell A_0(M_\ell^2) \left(\beta_\ell^{(0)} q^0 + \beta_\ell^{(2)} q^2 + \beta_\ell^{(4)} q^4 \right) \\ & + \left(\gamma^{(0)} q^0 + \gamma^{(2)} q^2 + \gamma^{(4)} q^4 \right) + \mathcal{O}\left(\frac{1}{q^2}\right), \quad (19) \end{aligned}$$

where the sum in n extends to all pairs of virtual mesons with masses M_i, M_j that occur in the loops, and $\lambda_n^{(2k)}$, $\beta_\ell^{(2k)}$ and $\gamma^{(2k)}$ are combinations of resonance parameters. The superindex $(2k)$ refers to the order at large- q^2 of the corresponding term. If we further expand the scalar functions around $q^2 = \infty$, the one-loop amplitude shows the general analytical structure

$$\begin{aligned} \Pi^{1-\text{loop}}(q^2) = & \left(\hat{\lambda}^{(0)} q^0 + \hat{\lambda}^{(2)} q^2 + \hat{\lambda}^{(4)} q^4 \right) \ln \frac{-q^2}{M_R^2} \\ & + \left(\hat{\gamma}^{(0)} q^0 + \hat{\gamma}^{(2)} q^2 + \hat{\gamma}^{(4)} q^4 \right) + \mathcal{O}\left(\frac{1}{q^2}\right), \quad (20) \end{aligned}$$

with M_R some arbitrary mass scale chosen to make the argument of the logarithms dimensionless. Note that the logarithmic part contains the absorptive contributions which define the spectral function $\text{Im } \Pi(q^2)$.

Local terms from \mathcal{L}^{GB} also contribute to the correlators through a polynomial in the $\tilde{L}_i, \tilde{C}_i \dots$ couplings:

$$\Pi^{\text{GB}}(q^2) = \tilde{L}_J + \tilde{C}_J q^2 + \dots \quad (21)$$

where the $\tilde{L}_J, \tilde{C}_J \dots$ refer to corresponding LECs or combination of them for the amplitude under consideration. As mentioned in Section II, the relevant $\mathcal{O}(p^4)$ couplings for the correlators in the chiral limit are \tilde{L}_{6-8} and \tilde{L}_{10} . The \mathcal{L}^{GB} couplings in Eq. (21) should be understood as bare parameters, which absorb the local ultraviolet divergences that may be contained in the $\hat{\gamma}^{(2k)}$.

The amplitudes for the linear combinations of correlators $SS - PP$ and $VV - AA$ also have the form shown in Eqs. (19, 20). These correlators are particularly useful for the purposes of determining the LECs, since we know they must vanish for $q^2 \rightarrow \infty$, as dictated by perturbative QCD [25]. This requirement translates into conditions on the terms shown in Eq. (20), which have the wrong short-distance behaviour. Due to their different analytical structure, the cancellations must occur separately for the logarithmic and polynomial parts. The vanishing of the non-polynomial part requires that $\hat{\lambda}^{(2k)} = 0$. The cancellation of the remaining polynomial is then achieved by tuning the local contributions from \mathcal{L}^{GB} to fulfill the equations

$$\tilde{L}_J + \hat{\gamma}^{(0)} = 0, \quad \tilde{C}_J + \hat{\gamma}^{(2)} = 0, \quad \dots \quad (22)$$

These constraints fix the value of the corresponding \mathcal{L}^{GB} couplings that contribute to the $VV - AA$ and $SS - PP$ correlators. Thus, the piece with the wrong high-energy behaviour disappears from the calculation and becomes irrelevant for the matching with χPT at low-energies.

Let us mention that a more restrictive condition than $\hat{\lambda}^{(2k)} = 0$ has been used in the literature [20, 21] to enforce the right short-distance behaviour. The spectral function $\text{Im } \Pi(q^2)$ associated to the SS, PP, VV and AA correlators are the sum of absorptive contributions corresponding to the different intermediate states,

$$\text{Im } \Pi(q^2) = \sum_n \text{Im } \Pi_n(q^2). \quad (23)$$

At one-loop, any of the possible absorptive contributions, n , comes from the two-particle cuts in the diagrams of Fig. 1. At large q^2 the vector and axial-vector spectral functions tend to a constant whereas the scalar and pseudoscalar ones grow like q^2 [25]. Therefore, since there is an infinite number of possible states, the absorptive contribution in the spin-1 correlators coming from each intermediate state should vanish in the $q^2 \rightarrow \infty$ limit if we assume a similar short-distance behavior for all of them. The high-energy behavior of the spin-0 spectral functions is not so clear as, *a priori*, a constant behavior for each intermediate cut does not seem to be excluded. However, the fact that $\Pi_{SS}(q^2) - \Pi_{PP}(q^2)$ vanishes as q^{-4} [25], the Brodsky-Lepage rules for the form factors [28] and the q^{-2} behavior of each one-particle intermediate cut seems to point out that every absorptive contribution to $\text{Im } \Pi(q^2)$ must also vanish at large momentum transfer. This assumption translates into $\lambda_n^{(2k)} = 0$ in Eq. (19), which is a particular solution of the more general condition $\hat{\lambda}^{(2k)} = 0$ pointed out above.

B. Pion form factors

We now turn to the analysis of the saturation condition for the LECs related to the scalar form factor (L_4 and L_5 at $\mathcal{O}(p^4)$), and with the vector form factor of the pion (L_9). The tree-level expression of the pion form factors behaves now as $\mathcal{O}(q^0)$ at large energies, while one-loop corrections given by the $\text{R}\chi\text{T}$ lagrangian are up to $\mathcal{O}(q^8)$. The allowed topologies with absorptive two-meson cuts are shown in Fig. 2. The main difference with respect to the analysis for the two-point correlators comes from the presence of triangle graphs. After the reduction of the one-loop diagrams to scalar functions, the form factor in the $q^2 \rightarrow \infty$ expansion reads

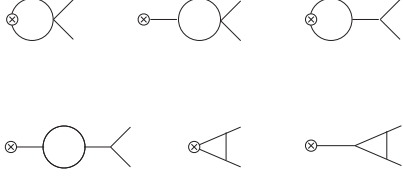


FIG. 2: Topologies in the one-loop pion form factor with a two-meson absorptive cut.

$$\begin{aligned}
\mathcal{F}^{1-\text{loop}}(q^2) = & \sum_n \sum_\ell C_0(q^2, 0, 0, M_i^2, M_j^2, M_\ell^2) \left(\kappa_{n,\ell}^{(2)} q^4 \right) \\
& + \sum_n B_0(q^2, M_i^2, M_j^2) \left(\lambda_n^{(2)} q^2 + \lambda_n^{(4)} q^4 + \lambda_n^{(6)} q^6 + \lambda_n^{(8)} q^8 \right) \\
& + \sum_\ell A_0(M_\ell^2) \left(\beta_\ell^{(2)} q^2 + \beta_\ell^{(4)} q^4 + \beta_\ell^{(6)} q^6 + \beta_\ell^{(8)} q^8 \right) \\
& + \left(\gamma^{(2)} q^2 + \gamma^{(4)} q^4 + \gamma^{(6)} q^6 + \gamma^{(8)} q^8 \right) + \mathcal{O}(q^0). \quad (24)
\end{aligned}$$

The sum over n extends to every s -channel two-particle cut in the diagrams of Fig. 2 involving mesons M_i and M_j . The three-point functions C_0 are generated from the triangle diagrams with internal masses $M_{i,j,\ell}$, where the subindex ' ℓ ' labels the virtual meson connecting the two outgoing pions. As before, the quantities $\kappa_{n,\ell}^{(2)}$, $\lambda_n^{(2k)}$, $\beta_\ell^{(2k)}$ and $\gamma^{(2k)}$ are combinations of resonance parameters (which obviously differ from those in Eq. (19), although we are using the same notation). It is straightforward to check that terms of order q^4 , q^6 and q^8 proportional to C_0 do not show up if we stick to resonance interaction terms with a chiral tensor of at most $\mathcal{O}(p^2)$. Note that since C_0 behaves asymptotically as $1/q^2$, the term $\kappa_{n,\ell}^{(2)}$ is of $\mathcal{O}(q^2)$ for large q^2 .

The expansion of the scalar functions at high energies leads to the analytic structure

$$\begin{aligned}
\mathcal{F}^{1-\text{loop}}(q^2) = & \left(\hat{\kappa}^{(2)} q^2 \right) \ln^2 \frac{-q^2}{M_R^2} \\
& + \left(\hat{\lambda}^{(2)} q^2 + \hat{\lambda}^{(4)} q^4 + \hat{\lambda}^{(6)} q^6 + \hat{\lambda}^{(8)} q^8 \right) \ln \frac{-q^2}{M_R^2} \\
& + \left(\hat{\gamma}^{(2)} q^2 + \hat{\gamma}^{(4)} q^4 + \hat{\gamma}^{(6)} q^6 + \hat{\gamma}^{(8)} q^8 \right) + \mathcal{O}(q^0), \quad (25)
\end{aligned}$$

with M_R some arbitrary mass scale chosen to make the argument of the logarithms dimensionless. The log squared terms arise from the C_0 functions.

The contributions to the form factor from \mathcal{L}^{GB} start now at $\mathcal{O}(q^2)$:

$$\mathcal{F}^{\text{GB}}(q^2) = \frac{\tilde{L}_J q^2}{F^2} + \frac{\tilde{C}_J q^4}{F^2} + \dots, \quad (26)$$

where, again, \tilde{L}_J and \tilde{C}_J refer to the corresponding combination of the $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ LECs, respectively. As

an example, for the vector form factor one has $\tilde{L}_J = 2\tilde{L}_9$ and $\tilde{C}_J = 4\tilde{C}_{90} - 4\tilde{C}_{88}$.

Based on the Brodsky-Lepage rules for the form factors of QCD currents [28], and on the large- q^2 behaviour of the spin-0 and spin-1 current correlators ($\mathcal{O}(q^2)$ for the spin-0 and $\mathcal{O}(q^0)$ for the spin-1 case) obtained from perturbative QCD, we can expect that the pion form factors behave at worst as a constant for $q^2 \rightarrow \infty$. This short-distance constraint requires that the terms shown in Eqs. (25, 26) vanish when put together. The vanishing of the logarithmic terms implies the constraints $\hat{\lambda}^{(2k)} = \hat{\kappa}^{(2k)} = 0$, and already lead to a well behaved spectral function $\text{Im}\mathcal{F}(q^2)$.

As a result, only polynomial terms remain in Eq. (25), which must be canceled with the local terms in Eq. (26) by a suitable choice of the couplings of \mathcal{L}^{GB} . The procedure leads to equations analogous to those shown in Eq. (22). In this way, the expressions for the R χ T vector and scalar pion form factors with the proper short-distance behaviour do not longer depend on the \tilde{L}_i , $\tilde{C}_i \dots$ couplings. Thus, the related χ PT couplings (L_4 , L_5 and L_9 at $\mathcal{O}(p^4)$) get determined in terms of the resonance parameters at NLO in $1/N_C$ through a matching with the R χ T result at low-energies.

As in the case of the two-current correlators, it is possible to consider the more stringent constraints which follow from requiring that each individual contribution to the spectral function $\text{Im}\mathcal{F}(q^2)$ vanish at high energies. This is achieved by the conditions $\lambda_n^{(2k)} = \sum_\ell \kappa_{n,\ell}^{(2k)} = 0$ for any s -channel absorptive cut n .

We want to emphasize that Sections IV A and IV B generalize the results of Ref. [22] to the case when both spin-0 and spin-1 resonance fields are present in R χ T.

C. Goldstone boson scattering

The $2 \rightarrow 2$ Goldstone boson scattering amplitude allows us to determine the $\mathcal{O}(p^4)$ χ PT couplings L_1 , L_2 and L_3 . The tree-level expression behaves as $\mathcal{O}(q^2)$ at large energies, while one-loop corrections can be up to $\mathcal{O}(q^{12})$ if spin-1 mesons are included in the theory. The relevant one-loop topologies are given in Fig. 3.

The study of the Goldstone scattering is more involved because the amplitudes depend in this case on two kinematic invariants, usually chosen among the Mandelstam variables s, t, u . In order to obtain useful constraints from high-energies, it is convenient to consider $s \leftrightarrow u$ symmetric amplitudes $T(\nu, t)$, with $\nu \equiv (s - u)/2$ [8, 26]. In that case, the forward scattering amplitude $T(\nu, t = 0)$ must obey a once-subtracted dispersion relation

$$T(\nu, 0) = T(0, 0) + \frac{\nu^2}{\pi} \int_0^\infty \frac{d\nu'^2}{\nu'^2} \frac{\text{Im}T(\nu', 0)}{(\nu'^2 - \nu^2)}, \quad (27)$$

with $\nu = s = -u$ for $t = 0$. Thus, at high energies one finds the behaviour $T(\nu \rightarrow \infty, 0) \sim \nu^0$. Note that $T(\nu, 0)$ can only depend on ν^2 for $s \leftrightarrow u$ symmetric amplitudes.

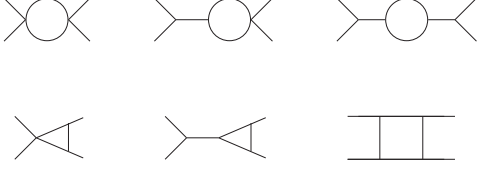


FIG. 3: Topologies in the one-loop scattering amplitude with two-meson cuts in the s , t or u -channels. The possible permutations in the labeling of the external legs are implicitly assumed.

The scattering amplitude calculated at one-loop within $\text{R}\chi\text{T}$ contains scalar integrals with up to four propagators. A given one-loop diagram thus has the general form

$$\begin{aligned}
T^{1\text{-loop}}(s, t, u) \sim & a(s, t, u)A_0(M_R) \\
& + b_s(s, t, u)B_0(s) + b_t(s, t, u)B_0(t) + b_u(s, t, u)B_0(u) \\
& + c_s(s, t, u)C_0(0, 0, s) + c_t(s, t, u)C_0(0, 0, t) \\
& + c_u(s, t, u)C_0(0, 0, u) \\
& + d_{st}(s, t, u)D_0(s, t) + d_{tu}(s, t, u)D_0(t, u) \\
& + d_{us}(s, t, u)D_0(u, s),
\end{aligned} \tag{28}$$

where we have shortened the notation by omitting the mass dependences of the scalar functions B_0 , C_0 and D_0 . Likewise, the dependence on the external leg momenta ($p_i^2 = 0$) is assumed implicitly. The factors $a(s, t, u)$, $b(s, t, u)$, $c(s, t, u)$ and $d(s, t, u)$ are rational functions with at most resonance double poles which depend on the structure of the meson vertices. If we consider the $s \leftrightarrow u$ symmetric amplitude with $t = 0$, the large- ν expansion of the one-loop scattering amplitude shows the general structure

$$\begin{aligned}
T^{1\text{-loop}}(\nu, 0) = & \left(\hat{\kappa}^{(4)}\nu^2 + \hat{\kappa}^{(8)}\nu^4 + \hat{\kappa}^{(12)}\nu^6 \right) \ln^2 \frac{-\nu^2}{M_R^4} \\
& + \left(\hat{\lambda}^{(4)}\nu^2 + \hat{\lambda}^{(8)}\nu^4 + \hat{\lambda}^{(12)}\nu^6 \right) \ln \frac{-\nu^2}{M_R^4} \\
& + \left(\hat{\gamma}^{(4)}\nu^2 + \hat{\gamma}^{(8)}\nu^4 + \hat{\gamma}^{(12)}\nu^6 \right) + \mathcal{O}(\nu^0).
\end{aligned} \tag{29}$$

It can be shown that for the case of the forward scattering the $\ln^2(-\nu^2/M_R^4)$ terms only arise from the three-point scalar functions. Local terms from the operators in \mathcal{L}^{GB} also contribute to $T(\nu, 0)$,

$$T^{\text{GB}}(\nu, 0) = \frac{\tilde{L}_J \nu^2}{F^4} + \dots \tag{30}$$

with \tilde{L}_J the $\mathcal{O}(p^4)$ coupling or combination of them for the corresponding scattering amplitude. For instance, $\tilde{L}_J = 8(2\tilde{L}_1 + 2\tilde{L}_2 + \tilde{L}_3)$ for the $\pi^0\pi^0 \rightarrow \pi^0\pi^0$ channel in the chiral limit.

In order to recover the proper short-distance behavior, $T(\nu \rightarrow \infty, 0) \sim \nu^0$, one needs that the bad behaved logarithmic terms get canceled, which is satisfied by the conditions $\hat{\lambda}^{(2k)} = \hat{\kappa}^{(2k)} = 0$. The polynomial pieces

can be further made to vanish by establishing relations among the $\hat{\gamma}^{(2k)}$ coefficients and the couplings \tilde{L}_J similar to Eqs. (22). Hence, only the well-behaved part of the amplitude, which does not depend on the \mathcal{L}^{GB} couplings, determines the scattering amplitude, in agreement with the requirements of resonance saturation.

More stringent constraints may also be considered for the scattering amplitude. The absorptive part $\text{Im} T$ is given by a sum of positive contributions coming from every possible 2-particle cut in the s -channel. Since there is an infinite number of intermediate states, it seems reasonable to expect that the contribution of each of them should vanish at high energies in order to have a $\mathcal{O}(\nu^0)$ behaviour for the total absorptive part.

V. SUMMARY

Resonance Chiral Theory supplemented with large- N_C arguments to rule its perturbative expansion provides a framework to obtain predictions for the low-energy constants of χPT in a systematic way. Resonance saturation within this formalism can be defined precisely: it states that the χPT LECs can be written in terms of only the resonance couplings and masses. The statement is not trivially satisfied because the $\text{R}\chi\text{T}$ amplitudes also depend on the parameters $\tilde{L}_i, \tilde{C}_i, \dots$ of the Goldstone boson sector which describes the self-interactions of the Goldstone bosons in the presence of resonances.

In this work we have proved that the saturation of the LECs holds at NLO in $1/N_C$, *i.e.* including one-loop corrections in $\text{R}\chi\text{T}$. Through an analysis of the analytic structure of the two-point correlators, the pion form-factors and the Goldstone scattering amplitude we have shown that the values of the parameters $\tilde{L}_i, \tilde{C}_i, \dots$ can be fixed once we enforce the QCD short-distance behaviour to the corresponding amplitudes. The resulting $\text{R}\chi\text{T}$ amplitudes get then written only in terms of parameters related to the dynamics of the resonances, and their low-energy expansion yields predictions for the LECs. The matching between $\text{R}\chi\text{T}$ and χPT is performed at the one-loop level in both theories, thus ensuring the cancellation of the chiral logs. Therefore the NLO estimates for the LECs obtained in this way do not suffer from the renormalization scale uncertainties inherent to the tree-level predictions.

The paper provides further insight on some of the conceptual aspects which have to be addressed in order to devise a consistent resonance effective lagrangian at the NLO in $1/N_C$, thus continuing the work initiated in Refs. [19, 22, 23].

Acknowledgments

We want to thank P. Masjuan, S. Peris, A. Pich and J. Portolés for their comments on the manuscript. This

work is supported in part by the Universidad CEU Cardenal Herrera (grant PRCEU-UCH20/08), by the Generalitat de Catalunya (grant SGR2005-00916), by the Spanish Government (FPA2007-60323, FPA2008-01430, the Juan de la Cierva program and Consolider-Ingenio

2010 CSD2007-00042, CPAN) and by the EU MRTN-CT-2006-035482 (FLAVIANet). P.R. acknowledges support from the DFG Sonderforschungsbereich/Transregio 9 “Computergestützte Theoretische Teilchenphysik”.

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- [1] S. Weinberg, *Physica A* **96** (1979) 327; J. Gasser and H. Leutwyler, *Annals Phys.* **158** (1984) 142; *Nucl. Phys. B* **250** (1985) 465; *Nucl. Phys. B* **250** (1985) 517; *Nucl. Phys. B* **250** (1985) 539.
 - [2] G. Ecker, *Prog. Part. Nucl. Phys.* **35** (1995) 1 [arXiv:hep-ph/9501357]; A. Pich, *Rept. Prog. Phys.* **58** (1995) 563 [arXiv:hep-ph/9502366].
 - [3] J. Bijnens, G. Colangelo and G. Ecker, *JHEP* **9902** (1999) 020 [arXiv:hep-ph/9902437]; *Annals Phys.* **280** (2000) 100 [arXiv:hep-ph/9907333].
 - [4] J. Bijnens, *Prog. Part. Nucl. Phys.* **58** (2007) 521 [arXiv:hep-ph/0604043].
 - [5] S. Necco, arXiv:0901.4257 [hep-lat].
 - [6] M. Gonzalez-Alonso, A. Pich and J. Prades, *Phys. Rev. D* **78** (2008) 116012 [arXiv:0810.0760 [hep-ph]].
 - [7] G. Ecker, J. Gasser, A. Pich and E. de Rafael, *Nucl. Phys. B* **321** (1989) 311.
 - [8] G. Ecker, J. Gasser, H. Leutwyler, A. Pich and E. de Rafael, *Phys. Lett. B* **223** (1989) 425.
 - [9] V. Cirigliano, G. Ecker, M. Eidemüller, R. Kaiser, A. Pich and J. Portolés, *Nucl. Phys. B* **753** (2006) 139 [arXiv:hep-ph/0603205].
 - [10] P. D. Ruiz-Femenía, A. Pich and J. Portolés, *JHEP* **0307** (2003) 003 [arXiv:hep-ph/0306157]; V. Cirigliano, G. Ecker, M. Eidemüller, A. Pich and J. Portolés, *Phys. Lett. B* **596** (2004) 96 [arXiv:hep-ph/0404004].
 - [11] G. 't Hooft, *Nucl. Phys. B* **72** (1974) 461; *Nucl. Phys. B* **75** (1974) 461; E. Witten, *Nucl. Phys. B* **160** (1979) 57.
 - [12] S. Peris, M. Perrottet and E. de Rafael, *JHEP* **9805** (1998) 011 [arXiv:hep-ph/9805442]; M. Knecht and E. de Rafael, *Phys. Lett. B* **424** (1998) 335 [arXiv:hep-ph/9712457].
 - [13] P. Masjuan and S. Peris, *JHEP* **0705** (2007) 040 [arXiv:0704.1247 [hep-ph]].
 - [14] B. Moussallam, *Phys. Rev. D* **51** (1995) 4939 [arXiv:hep-ph/9407402]; M. Knecht, S. Peris, M. Perrottet and E. de Rafael, *Phys. Rev. Lett.* **83** (1999) 5230 [arXiv:hep-ph/9908283]; M. Knecht and A. Nyffeler, *Eur. Phys. J. C* **21** (2001) 659 [arXiv:hep-ph/0106034]; V. Cirigliano, G. Ecker, M. Eidemüller, R. Kaiser, A. Pich and J. Portolés, *JHEP* **0504** (2005) 006 [arXiv:hep-ph/0503108].
 - [15] J. Bijnens, E. Gamiz, E. Lipartia and J. Prades, *JHEP* **0304** (2003) 055 [arXiv:hep-ph/0304222].
 - [16] J. J. Sanz-Cillero, *Phys. Lett. B* **649** (2007) 180 [arXiv:hep-ph/0702217].
 - [17] M. Golterman and S. Peris, *Phys. Rev. D* **74** (2006) 096002 [arXiv:hep-ph/0607152].
 - [18] O. Catà and S. Peris, *Phys. Rev. D* **65** (2002) 056014 [arXiv:hep-ph/0107062].
 - [19] I. Rosell, J. J. Sanz-Cillero and A. Pich, *JHEP* **0408** (2004) 042 [arXiv:hep-ph/0407240].
 - [20] I. Rosell, J. J. Sanz-Cillero and A. Pich, *JHEP* **0701** (2007) 039 [arXiv:hep-ph/0610290].
 - [21] A. Pich, I. Rosell and J. J. Sanz-Cillero, *JHEP* **0807** (2008) 014 [arXiv:0803.1567 [hep-ph]].
 - [22] J. Portolés, I. Rosell and P. Ruiz-Femenía, *Phys. Rev. D* **75** (2007) 114011 [arXiv:hep-ph/0611375].
 - [23] I. Rosell, P. Ruiz-Femenía and J. Portolés, *JHEP* **0512** (2005) 020 [arXiv:hep-ph/0510041].
 - [24] A. Pich, arXiv:0812.2631 [hep-ph].
 - [25] S. Weinberg, *Phys. Rev. Lett.* **18** (1967) 507; M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, *Nucl. Phys. B* **147** (1979) 385.
 - [26] G. Ecker, *Acta Phys. Polon. B* **38** (2007) 2753-2762 [arXiv:hep-ph/0702263]; G. Ecker and C. Zauner, *Eur. Phys. J. C* **52** (2007) 315-323 [arXiv:0705.0624 [hep-ph]].
 - [27] G. Passarino and M. J. G. Veltman, *Nucl. Phys. B* **160** (1979) 151.
 - [28] G. P. Lepage and S. J. Brodsky, *Phys. Lett. B* **87** (1979) 359; *Phys. Rev. D* **22** (1980) 2157.